

On D-brane action at order α'^2

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Abstract

We use compatibility of D-brane action with linear T-duality, S-duality and with S-matrix elements as guiding principles to find all world volume couplings of one massless closed and two open strings at order α'^2 in type II superstring theories. In particular, we find that the squares of second fundamental form appear only in world volume curvatures, and confirm the observation that dilaton appears in string frame action via the transformation $\hat{R}_{\mu\nu} \rightarrow \hat{R}_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi$.

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1 Introduction and Results

The low energy effective field theory of D_p -branes in type II superstring theories consists of the Dirac-Born-Infeld (DBI) [1] and the Chern-Simons (CS) actions [2], *i.e.*,

$$S_p = S_p^{DBI} + S_p^{CS} \quad (1)$$

The curvature corrections to the DBI action have been found in [3] by requiring consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude [4, 5]. For totally-geodesic embedding of world-volume in ambient spacetime in which second fundamental form is zero, the corrections in string frame for zero B-field and for constant dilaton are³

$$S_p^{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[R_{abcd} R^{abcd} - 2\hat{R}_{ab} \hat{R}^{ab} - R_{abij} R^{abij} + 2\hat{R}_{ij} \hat{R}^{ij} \right] \quad (2)$$

where $\hat{R}_{ab} = \tilde{G}^{cd} R_{cadb}$, $\hat{R}_{ij} = \tilde{G}^{cd} R_{cidj}$ and $\tilde{G} = \det(\tilde{G}_{ab})$ where \tilde{G}_{ab} is the pull-back of bulk metric onto the world-volume, *i.e.*,

$$\tilde{G}_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} G_{\mu\nu}$$

The Riemann curvatures in (2) are the pull-back of the spacetime curvature onto tangent and normal bundles [3].

The curvature corrections to the CS part can be found by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action [6, 7, 8]. These corrections involve the quadratic order of the curvatures at order α'^2 . However, the consistency of the effective action with the S-matrix elements of one NSNS and one RR vertex operators requires the CS part at this order to have linear curvature corrections as well [9], *i.e.*,

$$S_p^{CS} \supset -\frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0 a_1 \dots a_p} \left[\frac{1}{(p+1)!} \nabla_j \mathcal{F}_{ia_0 \dots a_p}^{(p+2)} \hat{R}^{ij} + \frac{1}{2!p!} \nabla_a \mathcal{F}_{ija_1 \dots a_p}^{(p+2)} R_{a_0}{}^{aij} \right] \quad (3)$$

where \mathcal{F}^{n+1} is the field strength of the RR potential n -form. The S-matrix calculations produce also the couplings in the CS part which involve linear field strength of B-field [9] in which we are not interested in this paper.

For arbitrary embeddings, the couplings (2) have been extended in [3] to

$$S_p^{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[(R_T)_{abcd} (R_T)^{abcd} - 2(\hat{R}_T)_{ab} (\hat{R}_T)^{ab} - (R_N)_{abij} (R_N)^{abij} + 2\bar{R}_{ij} \bar{R}^{ij} \right] \quad (4)$$

³Our index convention is that the Greek letters (μ, ν, \dots) are the indices of the space-time coordinates, the Latin letters (a, d, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices.

where the world-volume curvature $(R_T)_{abcd}$ and $(R_N)^{abij}$ obey the Gauss-Codazzi equations, *i.e.*,

$$\begin{aligned}(R_T)_{abcd} &= R_{abcd} + \delta_{ij}(\Omega_{ac}{}^i \Omega_{bd}{}^j - \Omega_{ad}{}^i \Omega_{bc}{}^j) \\ (R_N)_{ab}{}^{ij} &= R_{ab}{}^{ij} + g^{cd}(\Omega_{ac}{}^i \Omega_{bd}{}^j - \Omega_{ac}{}^j \Omega_{bd}{}^i)\end{aligned}\tag{5}$$

where Ω_{ab}^i is the second fundamental form [3]⁴ The relation between $(\hat{R}_T)_{ab}$ and the world volume curvature is then

$$(\hat{R}_T)_{ab} = \hat{R}_{ab} + \delta_{ij}(\Omega_c{}^{ci} \Omega_{ab}{}^j - \Omega_{ca}{}^i \Omega_b{}^{cj})\tag{6}$$

In equation (4), $\bar{R}^{ij} = \hat{R}^{ij} + g^{ab}g^{cd}\Omega_{ac}{}^i \Omega_{bd}{}^j + \dots$ where dots stand for unknown terms which involve the trace of the second fundamental form. They could not be fixed in [3] because the couplings in [3] have been found by requiring the consistency of the corresponding couplings with the S-matrix element of one closed and two open string vertex operators for which the trace of the second fundamental form is zero. They may be fixed, however, by requiring the consistency of the couplings with dualities.

In static gauge and to the linear order of fields, the second fundamental form has the following simple form:

$$\Omega_{ab}{}^i = \partial_a \partial_b \chi^i + \Gamma_{ab}^i\tag{7}$$

where χ^i is the massless transverse scalar field and Γ_{ab}^i is the Levi-Civita connection. The couplings of one graviton and two transverse scalars in (4) have been shown to be consistent with the corresponding S-matrix elements [3]. However, there are couplings in (4) which involve the trace of the second fundamental form which can not be checked with the S-matrix element of one closed and two open string vertex operators. We will show, among other things, that the trace term in $(\hat{R}_T)_{ab}$ is required by the consistency of the couplings (4) with T-duality. Moreover, we will find that the duality fixes the dots in \bar{R}^{ij} to be

$$\bar{R}^{ij} = \hat{R}^{ij} + g^{ab}g^{cd}(\Omega_{ac}{}^i \Omega_{bd}{}^j - \Omega_{ab}{}^i \Omega_{cd}{}^j)\tag{8}$$

where the last term is the trace of the second fundamental form.

It has been observed in [10, 11] that the consistency of the closed string couplings with T-duality requires the couplings of non-constant dilaton appear in the world volume action via the transformation

$$\hat{R}_{ab} \rightarrow \mathcal{R}_{ab} = \hat{R}_{ab} + \partial_a \partial_b \Phi \quad \hat{R}_{ij} \rightarrow \mathcal{R}_{ij} = \hat{R}_{ij} + \partial_i \partial_j \Phi\tag{9}$$

We will find that the transformation of the couplings (4) under the above replacement produces the couplings of one dilaton and two transverse scalars which are consistent with

⁴Note that there is a minus sign typo on the right hand side of $(R_N)_{ab}{}^{ij}$ in reference [3]. For totally-geodesic embedding, $(R_N)_{ab}{}^{ij}$ must be equal $R_{ab}{}^{ij}$.

the dualities and with the corresponding S-matrix elements. In other words, the extension of the couplings (2) to include the curvature, the dilaton and the second fundamental form are

$$S_p^{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[(R_T)_{abcd} (R_T)^{abcd} - 2(\hat{\mathcal{R}}_T)_{ab} (\hat{\mathcal{R}}_T)^{ab} - (R_N)_{abij} (R_N)^{abij} + 2\bar{\mathcal{R}}_{ij} \bar{\mathcal{R}}^{ij} \right] \quad (10)$$

where $(\hat{\mathcal{R}}_T)_{ab}$ and $\bar{\mathcal{R}}_{ij}$ are the same as $(\hat{R}_T)_{ab}$ and \bar{R}_{ij} , respectively, in which the replacement (9) have been performed. We will show that similar extension exists for the couplings (3), *i.e.*, the consistency of the couplings with dualities and with the S-matrix requires the following extension of (3):

$$S_p^{CS} \supset -\frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0 a_1 \dots a_p} \left[\frac{1}{(p+1)!} \nabla_j \mathcal{F}_{ia_0 \dots a_p}^{(p+2)} \bar{\mathcal{R}}^{ij} + \frac{1}{2!p!} \nabla_a \mathcal{F}_{ij a_1 \dots a_p}^{(p+2)} (R_N)_{a_0}{}^{aij} \right] \quad (11)$$

The coupling of the RR field strength and dilaton in the first term above has been already shown in [10] to be consistent with the linear T-duality and with the S-matrix.

In general, one expects that the consistency of the world volume couplings with full non-linear T-duality and S-duality would fix all couplings at order α'^2 [12, 11], *e.g.*, the T-duality would relate the couplings (11) to the standard CS couplings $\mathcal{C}^{p-3}(R_T \wedge R_T - R_N \wedge R_N)$ at order α'^2 . They would involve also the world volume gauge field, the spacetime B-field and other RR-fields. In this paper, however, we will use only linear T-duality and S-duality. As a result, we will find many couplings which are consistent with such simplified dualities. We are interested in the couplings of one closed and two open string states in this paper. Even the coefficients of such couplings can not be fully fixed by the linear dualities. To reduce the number of arbitrary coefficients, we use consistency of the couplings with the corresponding S-matrix elements as well. This latter condition fixes all unknown coefficients of the couplings in the DBI part, *i.e.*, we will find the couplings (10) and the following couplings in the string frame:

$$S_p^{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[\mathcal{R}_{bd} (\partial_a F^{ab} \partial_c F^{cd} - \partial_a F_c{}^d \partial^c F^{ab}) + \frac{1}{2} R_{bdce} \partial^c F^{ab} \partial^e F_a{}^d + \frac{1}{4} \mathcal{R}_d{}^d (\partial_a F^{ab} \partial_c F_b{}^c + \partial_b F_a{}^c \partial_c F^{ab}) + \Omega_a{}^{ai} \partial_d H_c{}^d{}_i \partial_b F^{bc} - \Omega^{bai} \left(\partial_b F_a{}^c \partial_d H_c{}^d{}_i + \partial^d F_a{}^c \partial_i H_{bcd} - \frac{1}{2} \partial^d F_a{}^c \partial_c H_{bdi} \right) \right] \quad (12)$$

where the scalar curvature $\mathcal{R}_a{}^a \equiv \tilde{G}^{ab} \hat{R}_{ab} + 2\partial^a \partial_a \Phi$ is invariant under linear T-duality as the Ricci curvatures \mathcal{R}_{ab} and \mathcal{R}_{ij} in (9). The consistency of the couplings with the dualities and with the S-matrix elements fixes also the couplings in the CS part to be those in (11) and the following couplings in the string frame:

$$S_p^{CS} \supset \frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0 a_1 \dots a_p} \left[\frac{1}{2!(p-2)!} \partial^a F_{a_1 a_2} \partial_b F_{aa_0} \partial^b \mathcal{F}_{a_3 a_4 \dots a_p}^{(p-2)} \right] \quad (13)$$

$$\begin{aligned}
& -\frac{1}{(p-1)!}\Omega_{a_0}{}^{ai}\partial_a F_{ba_1}\partial^b\mathcal{F}_{ia_2a_3\cdots a_p}^{(p)} + \frac{1}{2!(p-1)!}\Omega^{bai}\partial_a F_{a_0a_1}\partial_b\mathcal{F}_{ia_2a_3\cdots a_p}^{(p)} \\
& -\frac{1}{2!(p-1)!}\Omega_a{}^{ai}\partial^b F_{a_0a_1}\partial_i\mathcal{F}_{ba_2a_3\cdots a_p}^{(p)} + \frac{1}{p!}\Omega_a{}^{ai}\partial^b F_{ba_0}\partial_i\mathcal{F}_{a_1a_2\cdots a_p}^{(p)} \\
& -\frac{1}{p!}\Omega^{bai}\partial_a F_{ba_0}\partial_i\mathcal{F}_{a_1a_2\cdots a_p}^{(p)} + \frac{1}{(p-1)!}\Omega_{a_0}{}^{ai}\partial^b F_{ba_1}\partial_i\mathcal{F}_{aa_2a_3\cdots a_p}^{(p)} \Big]
\end{aligned}$$

In the CS part, there is another multiplet whose coefficient can not be fixed by the linear dualities and by the S-matrix elements of one closed and two open strings. It involves, however, the square of the second fundamental form. On the other hand, as the couplings (10) and (11) indicate, the square of the second fundamental form combines with the appropriate curvatures to form world volume curvatures R_T and \bar{R} . Since the coefficients of the curvature terms are already fixed in (3), we expect the coefficient of this multiplet to be zero.

An outline of the paper is as follows: In the next section, we review the constraints that linear T-duality and S-duality may impose on an effective world volume action. In section 3, we review the contact terms of the S-matrix element of one closed and two open strings at order α'^2 . In section 4, we construct all coupling of one NSNS and two NS strings with arbitrary coefficients, and find the coefficients by requiring the consistency of the couplings with the linear dualities and with the S-matrix elements. In section 5, we construct all coupling of one RR and two NS strings with arbitrary coefficients, and find the coefficients by requiring the consistency of the couplings with the linear dualities and with the S-matrix elements.

2 Linear duality constraints

The T-duality and S-duality transformations on massless field are in general nonlinear. Constraining the effective actions to be invariant under these nonlinear transformations which may fix all couplings of bosonic fields including the non-perturbative effects [13], would be a difficult task (see [14, 12, 11] for nonlinear T-duality). In this paper, however, we are interested only in the world volume couplings of one massless closed and two open string states at order α'^2 . Using the fact that the world volume couplings of one closed string and the couplings of one closed and one open strings have no higher derivative corrections in the superstring theory, one realizes that the higher derivative couplings of one closed and two open string states must be invariant under linear duality transformations.

The full set of nonlinear T-duality transformations has been found in [15, 16, 17, 18, 19]. We consider a background consists of a constant dilaton ϕ_0 and a metric which is flat in all directions except the killing direction y which is a circle with radius ρ . Assuming quantum fields are small perturbations around this background, *e.g.*, $G_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ and $G_{yy} = \frac{\rho^2}{\alpha'}(1 + 2h_{yy})$ where $\mu, \nu \neq y$, the T-duality transformations for the background are $e^{2\phi_0} =$

$\frac{\alpha' e^{2\phi_0}}{\rho^2}, \tilde{G}_{\mu\nu} = \eta_{\mu\nu}$, $\tilde{G}_{yy} = \frac{\alpha'}{\rho^2}$ and the quantum fluctuations at the linear order take the following form⁵:

$$\begin{aligned} \tilde{\phi} &= \phi - \frac{1}{2} h_{yy}, \tilde{h}_{yy} = -h_{yy}, \tilde{h}_{\mu y} = B_{\mu y}, \tilde{B}_{\mu y} = h_{\mu y}, \tilde{h}_{\mu\nu} = h_{\mu\nu}, \tilde{B}_{\mu\nu} = B_{\mu\nu} \\ \tilde{C}_{\mu\dots\nu y}^{(n)} &= C_{\mu\dots\nu}^{(n-1)}, \tilde{C}_{\mu\dots\nu}^{(n)} = C_{\mu\dots\nu y}^{(n+1)} \end{aligned} \quad (14)$$

The T-duality transformation of the world volume gauge field when it is along the Killing direction, is $\tilde{A}_y = \chi_y$ where χ_y is the transverse scalar. Similarly, $\tilde{\chi}_y = A_y$. When the gauge field and the transverse scalar field are not along the Killing direction, they are invariant under the T-duality. We are interested in applying the above linear T-duality transformations on the quantum fluctuations and apply the full nonlinear T-duality on the background. The latter requires the CS part to have no overall dilaton factor and the DBI part to have the overall factor $e^{-\Phi} \sqrt{-\tilde{G}}$.

Following [20], the effective couplings which are invariant under the above linear T-duality can be constructed as follows: We first write, in the static gauge, all couplings on the world volume of D_p -brane involving one massless closed and two open string states, in terms of the world volume indices a, b, \dots and the transverse indices i, j, \dots . We call this action S_p . Then we reduce the action to the 9-dimensional space. It produces two different actions. In one of them, the Killing direction y is a world volume direction, *i.e.*, $a = (\tilde{a}, y)$, which we call it S_p^w , and in the other one the Killing direction y is a transverse direction, $i = (\tilde{i}, y)$, which we call it S_p^t . The transformation of S_p^w under the linear T-duality (14) which we call it S_{p-1}^{wT} , must be equal to S_{p-1}^t up to some total derivative terms, *i.e.*,

$$S_{p-1}^{wT} - S_{p-1}^t = 0 \quad (15)$$

This constrains the unknown coefficients in the original action S_p .

The S-duality of type IIB theory produces another set of constraints on the coefficients of S_p . Under the S-duality, the graviton in Einstein frame, *i.e.*, $G_{\mu\nu}^E = e^{-\Phi/2} G_{\mu\nu}$, the transverse scalar fields and the RR four-form are invariant, and the following objects transform as doublets [21, 22, 23]:

$$\begin{aligned} \mathcal{B} &\equiv \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \rightarrow (\Lambda^{-1})^T \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} \\ \mathcal{F} &\equiv \begin{pmatrix} *F \\ G(F) \end{pmatrix} \rightarrow (\Lambda^{-1})^T \begin{pmatrix} *F \\ G(F) \end{pmatrix} \end{aligned} \quad (16)$$

where the matrix $\Lambda \in SL(2, Z)$ and $G(F)$ is a nonlinear function of F, Φ, C . To the linear order of the quantum fluctuations and nonlinear background which we call it linear S-duality⁶, $G(F) = e^{-\phi_0} F$ where ϕ_0 is the constant dilaton background [21]. In above equation

⁵Note that if one considers full T-duality transformation for background and quantum fluctuations, then the effective action would contains all couplings at order α'^2 , *e.g.*, H^4 or $(\partial F)^2 H^2$. However, in this paper we are interested only in the couplings consisting of one closed and two open string fields, hence we consider only linear T-duality.

⁶Note that we consider finite $SL(2, Z)$ transformation but infinitesimal quantum fluctuations.

$(*F)_{ab} = \epsilon_{abcd}F^{cd}/2$. The transformation of the dilaton and the RR scalar C appears in the transformation of the $SL(2, Z)$ matrix \mathcal{M}

$$\mathcal{M} = e^\phi \begin{pmatrix} |\tau|^2 & C \\ C & 1 \end{pmatrix} \quad (17)$$

where $\tau = C + ie^{-\Phi}$. This matrix transforms as [21]

$$\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T \quad (18)$$

To the zeroth and the first order of quantum fluctuations and nonlinear order of the background field ϕ_0 , the matrix \mathcal{M} is

$$\mathcal{M}_0 = \begin{pmatrix} e^{-\phi_0} & 0 \\ 0 & e^{\phi_0} \end{pmatrix}, \quad \delta\mathcal{M} = \begin{pmatrix} -e^{-\phi_0}\phi & e^{\phi_0}C \\ e^{\phi_0}C & e^{\phi_0}\phi \end{pmatrix} \quad (19)$$

They transform as (18) under the $SL(2, Z)$ transformations.

Using the above transformations, it is obvious that there must be no couplings in the Einstein frame between one dilaton and two transverse scalars because it is impossible to construct $SL(2, Z)$ invariant from \mathcal{M}_0 and one $\delta\mathcal{M}$, *i.e.*, $\text{Tr}(\mathcal{M}_0^{-1}\delta\mathcal{M}) = 0$. This produces a set of constraint on the coefficients of the effective action S_p .

One can easily found that the following structures are invariant under the linear S-duality transformation:

$$\begin{aligned} \partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B} &= e^{-\phi_0}\partial F\partial^2 B - \partial(*F)\partial^2 C^{(2)} \\ \partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F} &= e^{-\phi_0}[\partial(*F)\partial(*F) + \partial F\partial F] \\ \partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F} &= e^{-\phi_0}\partial^2\Phi\partial F\partial F - e^{-\phi_0}\partial^2\Phi\partial(*F)\partial(*F) + \partial^2 C\partial F\partial(*F) + \partial^2 C\partial(*F)\partial F \end{aligned} \quad (20)$$

Up to total derivative terms then the couplings of one closed and two open string states on the world volume of D₃-brane should appear in the structures $R\Omega\Omega$, $\partial^2 C^{(4)}\Omega\Omega$, $\Omega\partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B}$, $R\partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F}$ and $\partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F}$ which are invariant under the linear S-duality. They constrain the coefficients of the couplings in S_p .

3 S-matrix constraints

Another set of constraints on the coefficients of S_p is produced by comparing the couplings with the S-matrix element of one closed and two open string states at order α'^2 . This S-matrix element has been calculated in [5]

$$A \sim \frac{\Gamma[-2t]}{\Gamma[1-t]^2} K(1, 2, 3) \quad (21)$$

where K is the kinematic factor and $t = -\alpha' k_1 \cdot k_2$ is the only Mandelstam variable in the amplitude. k_1 and k_2 are the open string momenta. The low energy expansion of the gamma functions is $\frac{\Gamma[-2t]}{\Gamma[1-t]^2} = -\frac{1}{2t} - \frac{\pi^2 t}{12} + \dots$. The first term produces the couplings which are consistent with the corresponding couplings in DBI and CS actions at order α'^0 [24]. The second term produces the following on-shell couplings in the Einstein frame when the closed string is NSNS state [24]:

$$\begin{aligned}
A(\chi, \chi, h) &\sim \left(2k_1 \cdot k_2 \zeta_1 \cdot \varepsilon_3 \cdot \zeta_2 + k_1 \cdot k_2 \zeta_1 \cdot \zeta_2 \varepsilon_{3a}^a + \zeta_1 \cdot p_3 \zeta_2 \cdot p_3 \varepsilon_{3a}^a \right. \\
&\quad \left. - 2k_1 \cdot \varepsilon_3 \cdot k_2 \zeta_1 \cdot \zeta_2 + 4\zeta_1 \cdot \varepsilon_3 \cdot k_1 \zeta_2 \cdot p_3 + (1 \longleftrightarrow 2) \right) (k_1 \cdot k_2)^2 \\
A(\chi, \chi, \phi) &\sim \frac{p-3}{2\sqrt{2}} \left(k_1 \cdot k_2 \zeta_1 \cdot \zeta_2 + \zeta_1 \cdot p_3 \zeta_2 \cdot p_3 + (1 \longleftrightarrow 2) \right) (k_1 \cdot k_2)^2 \\
A(\chi, a, b) &\sim -2i \left(2k_1^a \zeta_{1i} f_{2ab} \varepsilon_3^{bi} - \zeta_1 \cdot p_3 f_{2ab} \varepsilon_3^{ab} \right) (k_1 \cdot k_2)^2 \\
A(a, a, h) &\sim 2 \left(\varepsilon_{3ab} f_1^{ac} f_2^b{}_c - \frac{1}{4} f_{1ab} f_2^{ab} \varepsilon_{3a}^a + (1 \longleftrightarrow 2) \right) (k_1 \cdot k_2)^2 \\
A(a, a, \phi) &\sim -\frac{p-7}{4\sqrt{2}} \left(f_{1ab} f_2^{ab} + (1 \longleftrightarrow 2) \right) (k_1 \cdot k_2)^2
\end{aligned}$$

where ζ_1, ζ_2 are the polarizations of the open string states and ε_3 is the polarization of the closed string. For the RR state, the couplings in the momentum space are [24]

$$\begin{aligned}
A(\chi, \chi, c_{(p+1)}) &\sim -\frac{2}{(p+1)!} \left(\zeta_1 \cdot p_3 \zeta_2 \cdot p_3 \varepsilon_3^{a_0 \dots a_p} + 2(p+1) \zeta_1^i k_1^{a_0} \zeta_2 \cdot p_3 \varepsilon_{3i}^{a_1 \dots a_p} \right. \\
&\quad \left. + p(p+1) \zeta_1^i \zeta_2^j k_1^{a_0} k_2^{a_1} \varepsilon_{3ij}^{a_2 \dots a_p} \right) \epsilon_{a_0 \dots a_p}^v (k_1 \cdot k_2)^2 + (1 \longleftrightarrow 2) \\
A(\chi, a, c_{(p-1)}) &\sim -\frac{2}{(p-1)!} \left(\zeta_1 \cdot p_3 f_2^{a_0 a_1} \varepsilon_3^{a_2 \dots a_p} + (p-1) \zeta_1^i f_2^{a_0 a_1} k_1^{a_2} \varepsilon_{3i}^{a_3 \dots a_p} \right) \epsilon_{a_0 \dots a_p}^v (k_1 \cdot k_2)^2 \\
A(a, a, c_{(p-3)}) &\sim -\frac{1}{2(p-3)!} f_1^{a_0 a_1} f_2^{a_2 a_3} \varepsilon_3^{a_4 \dots a_p} \epsilon_{a_0 \dots a_p}^v (k_1 \cdot k_2)^2 + (1 \longleftrightarrow 2)
\end{aligned}$$

Compatibility of the couplings with above amplitudes constrains the coefficients in S_p .

It has been argued in [12] that to construct the effective action for probe branes, one has to impose the bulk equations of motion at order α'^0 into S_p . Since we are interested in the world volume couplings which have linear closed string fields, we have to impose the supergravity equations of motion at linear order, *i.e.*,

$$\begin{aligned}
R + 4\nabla^2 \Phi &= 0 \\
R_{\mu\nu} + 2\nabla_{\mu\nu} \Phi &= 0 \\
\nabla^\rho H_{\rho\mu\nu} &= 0 \\
\nabla^{\mu_1} \mathcal{F}_{\mu_1 \mu_2 \dots \mu_n}^{(n)} &= 0
\end{aligned} \tag{22}$$

where μ, ν, ρ are the bulk indices. Using these equations, one finds

$$\begin{aligned}
R_{\mu}{}^i{}_{\nu i} &= -2\nabla_{\mu\nu}\Phi - R_{\mu}{}^c{}_{\nu c} \\
\nabla^i{}_i\Phi &= -\nabla^a{}_a\Phi \\
\nabla^i H_{i\mu\nu} &= -\nabla^a H_{a\mu\nu} \\
\nabla^i \mathcal{F}_{i\mu_2\cdots\mu_n}^{(n)} &= -\nabla^a \mathcal{F}_{a\mu_2\cdots\mu_n}^{(n)}
\end{aligned} \tag{23}$$

which indicates that the terms on the left-hand side are not independent. In other words, the coefficients of the couplings in S_p which involve the terms on the left-hand side above must be zero.

4 DBI couplings

In this section, using the mathematica package “xAct” [25], we are going to write all couplings of one closed string NSNS state and two open strings with unknown coefficients. We then constrain the coefficients by imposing the consistency of the couplings with the linear dualities and with the corresponding S-matrix element. Since all such couplings are too many to be written them once, we consider the couplings with specific closed string NSNS state and open string NS states.

4.1 One graviton and two transverse scalar fields

We begin with the couplings of one graviton and two transverse scalar fields. The transverse scalar fields should appear in the action through the pull-back of bulk tensors, through the Taylor expansion of bulk tensors or through the second fundamental form. Since there is no higher-derivative correction to the couplings of one closed string and one open string in type II superstring theories, *e.g.*, there is no coupling with structure $DR\Omega$ or $RD\Omega$, the pull-back operator and Taylor expansion would produce no coupling between two scalars and one curvature from $DR\Omega$ or $RD\Omega$. Therefore, the only possibility for the two transverse scalars is through the second fundamental form. All such couplings at order α'^2 are the following:

$$\begin{aligned}
S_{h\chi\chi} = \frac{\pi^2\alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \Big[& w_1 R^{bc}{}_{bc} \Omega_a{}^{ai} \Omega_d{}^d{}_i + 2w_2 R^{bj}{}_{bj} \Omega_a{}^{ai} \Omega_c{}^c{}_i + w_3 R^{ij}{}_{ij} \Omega_a{}^{ak} \Omega_b{}^b{}_k \\
& + w_4 R^b{}_{ibj} \Omega_a{}^{ai} \Omega_c{}^c{}_j + w_5 R_i{}^j{}_{kj} \Omega_a{}^{ai} \Omega_b{}^{bk} + w_6 R^{bc}{}_{bc} \Omega_{dai} \Omega^{dai} \\
& + 2w_7 R^{bj}{}_{bj} \Omega_{cai} \Omega^{cai} + w_8 R^{kj}{}_{kj} \Omega_{bai} \Omega^{bai} + w_9 R^b{}_{ibj} \Omega_{ca}{}^j \Omega^{cai} \\
& + w_{10} R_i{}^j{}_{kj} \Omega_{ba}{}^k \Omega^{bai} + w_{11} R_d{}^c{}_{bc} \Omega^b{}_{ai} \Omega^{dai} + w_{12} R_c{}^j{}_{bj} \Omega^b{}_{ai} \Omega^{cai} \\
& + w_{13} R_{cbij} \Omega^b{}_a{}^j \Omega^{cai} + w_{14} R_{cibj} \Omega^b{}_a{}^j \Omega^{cai} - w_{15} R_{cjb i} \Omega^b{}_a{}^j \Omega^{cai} \\
& + w_{16} R_d{}^c{}_{bc} \Omega_a{}^{ai} \Omega^{bd}{}_i + w_{17} R_c{}^j{}_{bj} \Omega_a{}^{ai} \Omega^{bc}{}_i + w_{18} R_{cibj} \Omega_a{}^{ai} \Omega^{bcj} \\
& + w_{19} R_{abdc} \Omega^{cb}{}_i \Omega^{dai} \Big]
\end{aligned} \tag{24}$$

Where w_i with $i = 1, 2, \dots, 19$ are the unknown constants that must be determined by imposing various constraints.

All above couplings are not independent. In fact by applying the cyclic symmetry of the Riemann curvature, one can neglect some of the constants. For example, one finds the coupling in (24) with coefficient w_{13} , w_{14} and w_{15} are not independent, *i.e.*,

$$\begin{aligned} w_{13}R_{cbij}\Omega_a^b{}^j\Omega^{cai} + w_{14}R_{cibj}\Omega_a^b{}^j\Omega^{cai} - w_{15}R_{cjb i}\Omega_a^b{}^j\Omega^{cai} = \\ (w_{13} + w_{15})R_{cbij}\Omega_a^b{}^j\Omega^{cai} + (w_{14} - w_{15})R_{cibj}\Omega_a^b{}^j\Omega^{cai} \end{aligned} \quad (25)$$

So the coupling with coefficient w_{15} is not independent and may be ignored from the list (24) before imposing various constraints. Alternatively, one may keep all couplings in (24) and imposes the constraints to find appropriate relations between the coefficients and at the end imposes the cyclic symmetry. The final result of course must be identical in both methods. However, we find the latter method is easier to apply by computer so we do it in this paper. In fact after imposing the constraints, we write the Riemann curvature in terms of metric. Then all terms that are related by the cyclic symmetry would be canceled. So the coefficients of all such terms can easily be set to zero.

By comparing the above couplings with (4) we find $w_9 = 1, w_{11} = 1, w_{16} = -1$ and $w_{19} = 1$. These constraints are in fact the S-matrix constraints because the couplings in (4) are fixed in [3] by comparing them with the corresponding S-matrix elements. Furthermore, the constraint that the bulk equations of motion (23) have to be imposed on the brane couplings, fixes the coefficients $w_2 = w_3 = w_5 = w_7 = w_8 = w_{10} = w_{12} = w_{17} = 0$.

4.2 One graviton and two gauge fields

Under T-duality, the transverse scalar field along the Killing direction transforms to the gauge field, *i.e.*, Ω transform to ∂F . So consistency of the couplings (24) with T-duality requires the couplings of one graviton and two gauge fields to have structure $R\partial F\partial F$. All such couplings are the following:

$$\begin{aligned} S_{haa} = \frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \Big[& z_1 R^{cd}{}_{cd} \partial_a F^{ae} \partial_b F_e^b + 2z_2 R^{ci}{}_{ci} \partial_a F^{ad} \partial_b F_d^b \\ & + z_3 R^{ij}{}_{ij} \partial_a F^{ac} \partial_b F_c^b + z_4 R_e^d{}_{cd} \partial_a F^{ae} \partial_b F^{bc} + z_5 R_d^i{}_{ci} \partial_a F^{ad} \partial_b F^{bc} + z_6 R_e^d{}_{cd} \partial_a F_b^c \partial^b F^{ae} \\ & + z_7 R_d^i{}_{ci} \partial_a F_b^c \partial^b F^{ad} + z_8 R^{cd}{}_{cd} \partial^b F^{ae} \partial_e F_{ab} + 2z_9 R^{ci}{}_{ci} \partial^b F^{ad} \partial_d F_{ab} + z_{10} R^{ij}{}_{ij} \partial^b F^{ac} \partial_c F_{ab} \\ & + z_{11} R^{cd}{}_{cd} \partial_b F_{ae} \partial^b F^{ae} + 2z_{12} R^{ci}{}_{ci} \partial_b F_{ad} \partial^b F^{ad} + z_{13} R^{ij}{}_{ij} \partial_b F_{ac} \partial^b F^{ac} + z_{14} R_e^d{}_{cd} \partial_b F_a^c \partial^b F^{ae} \\ & + z_{15} R_d^i{}_{ci} \partial_b F_a^c \partial^b F^{ad} + z_{16} R_{aec d} \partial_b F^{cd} \partial^b F^{ae} + z_{17} R_{aced} \partial_b F^{cd} \partial^b F^{ae} + z_{18} R_b^d{}_{cd} \partial^b F^{ae} \partial^c F_{ae} \\ & + z_{19} R_b^i{}_{ci} \partial^b F^{ad} \partial^c F_{ad} + z_{20} R_e^d{}_{cd} \partial^b F^{ae} \partial^c F_{ab} + z_{21} R_d^i{}_{ci} \partial^b F^{ad} \partial^c F_{ab} + z_{22} R_b^d{}_{cd} \partial_a F^{ae} \partial^c F_e^b \\ & + z_{23} R_b^i{}_{ci} \partial_a F^{ad} \partial^c F_d^b + z_{24} R_{ebcd} \partial^b F^{ae} \partial^d F_a^c + z_{25} R_{ecbd} \partial^b F^{ae} \partial^d F_a^c + z_{26} R_{edbc} \partial^b F^{ae} \partial^d F_a^c \\ & + z_{27} R_{aec d} \partial^b F^{ae} \partial^d F_b^c + z_{28} R_{aced} \partial^b F^{ae} \partial^d F_b^c + z_{29} R_{ebcd} \partial_a F^{ae} \partial^d F^{bc} + z_{30} R_{edbc} \partial_a F^{ae} \partial^d F^{bc} \Big] \end{aligned} \quad (26)$$

Where z_i with $i = 1, 2, \dots, 30$ are constants that must be determined by imposing the constraints and F^{ab} is field strength of the gauge field. Here also one may impose the cyclic symmetry and the Bianchi identity $dF = 0$ before imposing the constraints to cancel some of the couplings in (26) before. However, we prefer to impose the cyclic symmetry and the bianchi identity after imposing the constraints. The bulk equations of motion (23) constrain $z_2 = z_3 = z_5 = z_7 = z_9 = z_{10} = z_{12} = z_{13} = z_{15} = z_{19} = z_{21} = z_{23} = 0$.

4.3 One dilaton and two transverse scalar fields

The same reason as in section 4.1, leads one to conclude that the couplings of one dilaton and two transverse scalar fields have structure $\partial\partial\Phi\Omega\Omega$. All such couplings are the following:

$$S_{\Phi\chi\chi} = \frac{\pi^2\alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[t_1 \Omega_a^a \Omega_{bi}^b \partial_c \partial^c \Phi + t_2 \Omega_a^a \Omega_{i}^{bc} \partial_c \partial_b \Phi + t_3 \Omega_a^c \Omega_i^{abi} \partial_c \partial_b \Phi \right. \\ \left. + t_4 \Omega_{abi} \Omega^{abi} \partial_c \partial^c \Phi + t_5 \Omega_a^a \Omega_{bi}^b \partial_j \partial^j \Phi + t_6 \Omega_{abi} \Omega^{abi} \partial_j \partial^j \Phi \right. \\ \left. + t_7 \Omega_a^a \Omega_b^b \partial_j \partial_i \Phi + t_8 \Omega_{ab} \Omega^{ab} \partial_j \partial_i \Phi \right] \quad (27)$$

Where t_i with $i = 1, 2, \dots, 8$ are the unknown constants that we must be determined. The bulk equations of motion (23) constrain $t_5 = t_6 = 0$.

4.4 One dilaton and two gauge fields

The consistency of the couplings (27) with T-duality requires the couplings of one dilaton and two gauge fields to have structure $\partial\partial\Phi\partial F\partial F$. All such couplings are the following:

$$S_{\Phi aa} = \frac{\pi^2\alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} \left[x_1 \partial_a F^{cd} \partial_b F_{cd} \partial^b \partial^a \Phi + x_2 \partial^b \partial^a \Phi \partial_c F_a^c \partial_d F_b^d \right. \\ \left. + x_3 \partial_a \partial^a \Phi \partial_b F^{bc} \partial_d F_c^d + x_4 \partial_b F_a^c \partial^b \partial^a \Phi \partial_d F_c^d + x_5 \partial_b F_{cd} \partial^b \partial^a \Phi \partial^d F_a^c \right. \\ \left. + x_6 \partial^b \partial^a \Phi \partial_c F_{bd} \partial^d F_a^c + x_7 \partial^b \partial^a \Phi \partial_d F_{bc} \partial^d F_a^c + x_8 \partial_a \partial^a \Phi \partial_c F_{bd} \partial^d F^{bc} \right. \\ \left. + x_9 \partial_a \partial^a \Phi \partial_d F_{bc} \partial^d F^{bc} + x_{10} \partial_a F^{ab} \partial_c F_b^c \partial_i \partial^i \Phi + x_{11} \partial_b F_{ac} \partial^c F^{ab} \partial_i \partial^i \Phi \right. \\ \left. + x_{12} \partial_c F_{ab} \partial^c F^{ab} \partial_i \partial^i \Phi \right] \quad (28)$$

where the constants x_i with $i = 1, 2, \dots, 12$ must be determined by imposing the constraints. The bulk equations of motion (23) constrain $x_{10} = x_{11} = x_{12} = 0$.

4.5 One B-field, one transverse scalar field and one gauge field

The final list of couplings in the DBI part is the couplings of one B-field, one transverse scalar field and one gauge field which is the following:

$$\begin{aligned}
S_{ba\chi} = \frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x e^{-\Phi} \sqrt{-\tilde{G}} & \left[\gamma_1 \Omega^{abi} \partial_a F^{cd} \partial_b H_{cdi} + \gamma_2 \Omega^{abi} \partial_b H_{adi} \partial_c F^{cd} \right. \\
& + \gamma_3 \Omega^{abi} \partial_a F^{cd} \partial_d H_{bci} + \gamma_4 \Omega^{abi} \partial_c F_a^c \partial_d H_b^d{}_i + \gamma_6 \Omega^{abi} \partial_b F_a^c \partial_d H_c^d{}_i \\
& + \gamma_5 \Omega_a^a{}^i \partial_b F^{bc} \partial_d H_c^d{}_i + \gamma_7 \Omega^{abi} \partial_b H_{cdi} \partial^d F_a^c + \gamma_8 \Omega^{abi} \partial_c H_{bdi} \partial^d F_a^c \\
& + \gamma_9 \Omega^{abi} \partial_d H_{bci} \partial^d F_a^c + \gamma_{10} \Omega_a^a{}^i \partial_c H_{bdi} \partial^d F^{bc} + \gamma_{11} \Omega_a^a{}^i \partial_d H_{bci} \partial^d F^{bc} \\
& + \gamma_{12} \Omega^{abi} \partial_a F^{cd} \partial_i H_{bcd} + \gamma_{13} \Omega^{abi} \partial^d F_a^c \partial_i H_{bcd} + \gamma_{14} \Omega_a^a{}^i \partial^d F^{bc} \partial_i H_{bcd} \\
& \left. - \gamma_{15} \Omega^{abi} \partial_c F_a^c \partial_j H_{bi}{}^j - \gamma_{17} \Omega^{abi} \partial_b F_a^c \partial_j H_{ci}{}^j - \gamma_{16} \Omega_a^a{}^i \partial_b F^{bc} \partial_j H_{ci}{}^j \right]
\end{aligned} \tag{29}$$

Where γ_i with $i = 1, 2, \dots, 17$ are the unknown constants. The equations of motion (23) fixes $\gamma_{15} = \gamma_{16} = \gamma_{17} = 0$.

We now consider the sum of the couplings in (24), (26), (27), (28) and (29), *i.e.*,

$$S_p^{DBI} = S_{h\chi\chi} + S_{haa} + S_{\Phi\chi\chi} + S_{\Phi aa} + S_{ba\chi} \tag{30}$$

and apply the T-duality constraint (15). It gives the following relations between the constants:

$$\begin{aligned}
t_8 = 1, \quad t_3 = -t_2, \quad t_1 = -\frac{t_2}{2} - \frac{x_4}{2} - x_3, \quad t_4 = \frac{t_2}{2} + \frac{x_4}{2} + x_8 - 2x_9 \\
t_7 = -1 - z_{14} - z_4 - z_6, \quad w_1 = \frac{1}{4} - \frac{x_3}{2} - \frac{x_4}{4}, \quad w_{15} = 2 + w_{14} - 2\gamma_1 + \gamma_3 + 2\gamma_6 + \gamma_7 - \gamma_9 \\
w_{18} = 2 + 2z_{14} - z_{29} + 2z_{30} + 2z_6, \quad w_4 = -1 - z_{14} - z_4 - z_6, \quad x_5 = 2x_1 - x_4 + x_7 \\
z_{20} = -z_{14} - 2z_{18} + z_{22}, \quad \gamma_{13} = w_{14} + 2\gamma_{12} + \gamma_3 + \gamma_6 - \gamma_9, \quad \gamma_5 = z_{14} + z_4 + z_6 - \gamma_6 \\
z_{28} = 2z_{14} + 4z_{16} + 2z_{17} + z_{24} - z_{26} - 2z_{27} + 2z_6 + 2\gamma_1 - \gamma_3 - 2\gamma_6 - \gamma_7 + \gamma_9 \\
\gamma_8 = \frac{1}{2} - \gamma_4 - \gamma_9, \quad w_6 = -\frac{1}{4} + \frac{x_4}{4} + \frac{x_8}{2} + x_9, \quad z_1 = \frac{x_3}{2} + \frac{x_4}{4} - \frac{z_{22}}{4}, \quad z_{25} = \frac{1}{2} + \frac{z_{22}}{2} - z_{26} \\
z_8 = \frac{x_4}{4} + \frac{x_8}{2} - \frac{z_{22}}{4} + x_9 - 2z_{11}, \quad \gamma_2 = -\frac{z_{29}}{2} + z_{14} + z_{30} + z_6 - \gamma_6 \\
\gamma_{11} = \frac{z_{14}}{2} - \frac{z_{29}}{4} + \frac{z_{30}}{2} + \frac{z_6}{2} - \frac{\gamma_{10}}{2} - \frac{\gamma_6}{2}, \quad x_6 = -x_2 - x_7 + z_{14} + z_4 + z_6
\end{aligned} \tag{31}$$

As can be seen, not all coefficients of the DBI part are fixed by imposing consistency of the couplings with the linear T-duality, so we need further constraints which may be the consistency with S-duality.

In general, S-duality connect the DBI couplings containing the NSNS states to the CS couplings containing RR states. However, the S-duality constrains even the couplings in the

DBI part. For example, the world volume couplings of D₃-brane in the Einstein frame must have no coupling with structure $\Phi\Omega\Omega$. This produces the following constraints:

$$\begin{aligned}\gamma_9 &= 2 - 2\gamma_1 + \gamma_3 + 2\gamma_6 + \gamma_7, \quad x_9 = \frac{1}{4} - \frac{x_4}{4} - \frac{x_8}{2} \\ z_4 &= \frac{1}{2} + x_3 + \frac{x_4}{2} - \frac{z_{29}}{2} + z_{30}, \quad z_6 = -1 - z_{14} - z_{30} + \frac{z_{29}}{2}\end{aligned}\tag{32}$$

Another constraint from the S-duality in the DBI part is that up to total derivative terms, the couplings of one graviton and two gauge fields in D₃-brane action must appear in the S-duality invariant structure $R\partial\mathcal{F}^T\mathcal{M}_0\partial\mathcal{F} = e^{-\phi_0}R(\partial(*F)\partial(*F) + \partial F\partial F)$. This produces the following constraints:

$$z_{14} = 0, \quad x_4 = 1 - 2x_3\tag{33}$$

The S-duality constrains the couplings of one dilaton and two gauge fields. It also connects them to the couplings of one RR scalar and two gauge fields. This is resulted from the fact that the S-duality invariant structure which contains the couplings of one dilaton and two gauge fields is $\partial\mathcal{F}^T\partial^2\mathcal{M}\partial\mathcal{F} = e^{-\phi_0}\partial^2\Phi(-\partial(*F)\partial(*F) + \partial F\partial F) + \dots$ where dots refer to the RR scalar couplings. This constraint on the couplings of one dilaton and two gauge fields produces the following relation:

$$x_8 = x_3$$

The S-duality connects the DBI couplings of one B-field, one gauge field and one transverse scalar field to the CS couplings of one RR two-form, one gauge field and one transverse scalar. In the next section we will write all couplings in the CS part and impose the T-duality condition (15). Then we will impose the above S-duality condition. It produces the following relation between the coefficients in the DBI part:

$$\gamma_6 = -1 + 2\gamma_1 - \gamma_3\tag{34}$$

and many relations between the coefficients in the CS part (see constraints in (43)).

Imposing the above relations between the coefficients in S_p^{DBI} , we find that the action (30) are consistent with the S-matrix elements in (22) except the following terms:

$$w_{14}\left(R_{bicj}\Omega^{bai}\Omega_a^c{}^j - R_{bjci}\Omega^{bai}\Omega_a^c{}^j + \Omega^{bai}\partial^d F_a^c \partial_i H_{bcd}\right)\tag{35}$$

They are not consistent with the couplings in (4) and with the corresponding S-matrix elements, so

$$w_{14} = 0\tag{36}$$

As can be seen, there are still many coefficients which are not fixed by the linear dualities and with the S-matrix elements.

We have considered all couplings in S_p^{DBI} which contains the Riemann curvature and the first derivative of the field strengths of the gauge field and the B-field. The Riemann curvature satisfies the cyclic symmetry and the field strengths satisfy the Bianchi identities. So we have to impose these symmetries in S_p^{DBI} . To perform this step, we write all field strengths in terms of their corresponding potentials and write the Riemann curvature in terms of

$$R_{abcd} = \partial_b \partial_c h_{ad} + \partial_a \partial_d h_{bc} - \partial_b \partial_d h_{ac} - \partial_a \partial_c h_{bd} \quad (37)$$

Then we find the coefficients $\gamma_1, \gamma_3, \gamma_7, \gamma_9, x_4, x_8, x_9, z_4, z_6, z_{14}$ disappear from the action. As a result, the couplings with the above coefficients represent only the cyclic symmetry and the Bianchi identities. So we ignore such terms in the DBI part. Finally, we find that the couplings with coefficients $\gamma_4, t_2, x_2, x_3, z_{22}, z_{29}, z_{30}$ are total derivative terms, so they can be eliminated from the DBI part too. The final result for the DBI part has no unknown coefficients! The couplings are those that appear in (10) and (12).

5 CS couplings

In this section, using the mathematica package “xAct” [25], we are going to write all couplings of one closed string RR state and two open NS strings with unknown coefficients. We then constrain the coefficients by imposing the consistency of the couplings with the linear dualities and with the corresponding S-matrix element. The S-matrix elements (22) indicates that the world volume couplings of D_p -brane in the CS part has three parts. One is the couplings of one C_{p-3} and two gauge fields, another one is the couplings of one C_{p-1} , one gauge field and one transverse scalar field, and the last one is the couplings of one C_{p+1} and two transverse scalar fields. Let us consider each case separately.

5.1 One RR and two gauge fields

In this section we construct all possible couplings of one C_{p-3} and two gauge fields. Using the bulk equations of motion (23), one finds there are 23 non-zero couplings, *i.e.*,

$$\begin{aligned} S_{caa} = & \frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1} x \epsilon^{a_0 a_1 \dots a_p} \left[\frac{1}{(p-2)!} \partial_a \mathcal{F}_{a_3 a_4 \dots a_p}^{(p-2)} \left(\frac{1}{2!} \kappa_1 \partial_b F_{a_1 a_2} \partial^b F^a{}_{a_0} + \kappa_2 \partial_{a_0} F^{ba} \partial_{a_2} F_{ba_1} \right. \right. \\ & + \kappa_3 \partial_{a_2} F_{ba_1} \partial^a F^b{}_{a_0} + \kappa_4 \partial_{a_2} F_{ba_1} \partial^b F^a{}_{a_0} + \frac{1}{2!} \kappa_5 \partial_{a_2} F_b{}^a \partial^b F_{a_0 a_1} \\ & + \frac{1}{2!} \kappa_6 \partial^a F_{ba_2} \partial^b F_{a_0 a_1} + \kappa_7 \partial_b F^b{}_{a_0} \partial_{a_2} F^a{}_{a_1} + \frac{1}{2!} \kappa_8 \partial_b F^b{}_{a_0} \partial^a F_{a_1 a_2} \Big) \\ & \left. + \frac{1}{(p-3)!} \partial^b \mathcal{F}_{ba_4 \dots a_p}^{(p-2)} \left(\frac{1}{2!} \frac{1}{2!} \kappa_9 \partial_a F_{a_2 a_3} \partial^a F_{a_0 a_1} \right. \right. \end{aligned}$$

$$\begin{aligned}
& +\kappa_{10}\partial_{a_1}F^a{}_{a_0}\partial_{a_3}F_{aa_2}+\frac{1}{2!}\kappa_{11}\partial^a F_{a_0a_1}\partial_{a_3}F_{aa_2}\Big) \\
& +\frac{1}{(p-3)!}\partial_a\mathcal{F}_{ba_4\cdots a_p}^{(p-2)}\Big(\kappa_{12}\partial_{a_1}F^b{}_{a_0}\partial_{a_3}F^a{}_{a_2}+\frac{1}{2!}\kappa_{13}\partial_{a_3}F^a{}_{a_2}\partial^b F_{a_0a_1} \\
& +\frac{1}{2!}\kappa_{14}\partial_{a_1}F^b{}_{a_0}\partial^a F_{a_2a_3}+\frac{1}{2!}\frac{1}{2!}\kappa_{15}\partial^b F_{a_0a_1}\partial^a F_{a_2a_3}\Big) \\
& +\frac{1}{(p-3)!}\partial_{a_4}\mathcal{F}_{aa_3a_5\cdots a_p}^{(p-2)}\Big(\frac{1}{2!}\kappa_{16}\partial_b F_{a_1a_2}\partial^b F^a{}_{a_0}+\kappa_{17}\partial_{a_2}F_{ba_1}\partial^b F^a{}_{a_0} \\
& +\kappa_{18}\partial_{a_0}F^{ba}\partial_{a_2}F_{ba_1}+\kappa_{19}\partial_{a_2}F_{ba_1}\partial^a F^b{}_{a_0}+\kappa_{20}\partial_b F^b{}_{a_0}\partial_{a_2}F^a{}_{a_1} \\
& +\frac{1}{2!}\kappa_{21}\partial_{a_2}F_b{}^a\partial^b F_{a_0a_1}+\frac{1}{2!}\kappa_{22}\partial^a F_{ba_2}\partial^b F_{a_0a_1}+\frac{1}{2!}\kappa_{23}\partial_b F^b{}_{a_0}\partial^a F_{a_1a_2}\Big)\Big]
\end{aligned} \tag{38}$$

where κ_i with $i = 1, \dots, 23$ are the unknown constants that have to be found. In above equation, $\mathcal{F}^{(p-2)}$ is the field strength of the RR potential C_{p-3} . One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the killing index y is a world volume index which is carried only by the RR field strength. When it is carried by the field strength of the gauge field, the consistency with T-duality requires the couplings of one C_{p-1} , one gauge field and one transverse scalar field which we consider them next.

5.2 One RR, gauge field and one transverse scalar field

All possible non-zero couplings of one RR potential C_{p-1} -form, one gauge field and one transverse scalar fields are the following:

$$\begin{aligned}
S_{ca\chi} = \frac{\pi^2\alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0a_1\cdots a_p} \Big[& \frac{1}{(p-1)!}\partial_b\mathcal{F}_{ia_2a_3\cdots a_p}^{(p)}\Big(\frac{1}{2!}\zeta_1\Omega^{bai}\partial_a F_{a_0a_1}+\zeta_2\Omega_{a_0}{}^{bi}\partial_a F^a{}_{a_1} \\
& +\zeta_3\Omega_{a_0}{}^{ai}\partial_a F^b{}_{a_1}+\zeta_4\Omega^{bai}\partial_{a_1}F_{aa_0}+\zeta_5\Omega_{a_0}{}^{ai}\partial_{a_1}F_a{}^b+\zeta_6\Omega_{a_0}{}^{ai}\partial^b F_{aa_1} \\
& +\zeta_7\Omega_a{}^{ai}\partial_{a_1}F^b{}_{a_0}+\frac{1}{2!}\zeta_8\Omega_a{}^{ai}\partial^b F_{a_0a_1}\Big) \\
& +\frac{1}{(p-2)!}\partial^b\mathcal{F}^{(p)}{}_{iba_3a_4\cdots a_p}\Big(\frac{1}{2!}\zeta_9\Omega_{a_0}{}^{ai}\partial_a F_{a_1a_2}+\zeta_{10}\Omega_{a_0}{}^{ai}\partial_{a_2}F_{aa_1}\Big) \\
& +\frac{1}{(p-2)!}\partial_b\mathcal{F}_{iaa_3a_4\cdots a_p}^{(p)}\Big(\zeta_{11}\Omega_{a_0}{}^{bi}\partial_{a_2}F^a{}_{a_1}+\frac{1}{2!}\zeta_{12}\Omega_{a_0}{}^{bi}\partial^a F_{a_1a_2} \\
& +\zeta_{13}\Omega_{a_0}{}^{ai}\partial_{a_2}F^b{}_{a_1}+\frac{1}{2!}\zeta_{14}\Omega_{a_0}{}^{ai}\partial^b F_{a_1a_2}\Big) \\
& +\frac{1}{(p-3)!}\partial_{a_4}\mathcal{F}_{iabaa_3a_5\cdots a_p}^{(p)}\Big(\zeta_{15}\Omega_{a_0}{}^{ai}\partial_{a_2}F^b{}_{a_1}+\frac{1}{2!}\zeta_{16}\Omega_{a_0}{}^{ai}\partial^b F_{a_1a_2}\Big) \\
& +\frac{1}{(p-2)!}\partial_{a_4}\mathcal{F}_{iba_2a_3a_5\cdots a_p}^{(p)}\Big(\zeta_{17}\Omega_{a_0}{}^{bi}\partial_a F^a{}_{a_1}+\frac{1}{2!}\zeta_{18}\Omega^{bai}\partial_a F_{a_0a_1} \\
& +\zeta_{19}\Omega^{bai}\partial_{a_1}F_{aa_0}+\zeta_{20}\Omega_a{}^{ai}\partial_{a_1}F^b{}_{a_0}\zeta_{21}\Omega_{a_0}{}^{ai}\partial_a F^b{}_{a_1}+\zeta_{22}\Omega_{a_0}{}^{ai}\partial_{a_1}F_a{}^b
\end{aligned}$$

$$\begin{aligned}
& +\zeta_{23}\Omega_{a_0}{}^{ai}\partial^b F_{aa_1} + \frac{1}{2!}\zeta_{24}\Omega_a{}^{ai}\partial^b F_{a_0a_1} \Big) \\
& + \frac{1}{(p-1)!}\partial_{a_4}\mathcal{F}_{ia_1a_2a_3a_5\cdots a_p}^{(p)} \Big(\zeta_{25}\Omega_a{}^{ai}\partial_b F_{a_0}^b + \zeta_{26}\Omega^{bai}\partial_b F_{aa_0} \\
& + \zeta_{27}\Omega_{a_0}{}^{bi}\partial_a F_b{}^a \Big) \\
& + \frac{1}{(p-2)!}\partial_i\mathcal{F}_{aba_3a_4\cdots a_p}^{(p)} \Big(\zeta_{28}\Omega_{a_0}{}^{ai}\partial_{a_2} F_{a_1}^b + \zeta_{29}\Omega_{a_0}{}^{ai}\partial^b F_{a_1a_2} \Big) \\
& + \frac{1}{(p-1)!}\partial_i\mathcal{F}_{ba_2a_3a_4\cdots a_p}^{(p)} \Big(\zeta_{30}\Omega_{a_0}{}^{bi}\partial_a F_{a_1}^a + \zeta_{31}\Omega^{bai}\partial_a F_{a_0a_1} \\
& + \zeta_{32}\Omega^{bai}\partial_{a_1} F_{aa_0} + \zeta_{33}\Omega_a{}^{ai}\partial_{a_1} F_{a_0}^b + \zeta_{34}\Omega_{a_0}{}^{ai}\partial_a F_{a_1}^b + \zeta_{35}\Omega_{a_0}{}^{ai}\partial_{a_1} F_a{}^b \\
& + \zeta_{36}\Omega_{a_0}{}^{ai}\partial^b F_{aa_1} + \frac{1}{2!}\zeta_{37}\Omega_a{}^{ai}\partial^b F_{a_0a_1} \Big) \\
& + \frac{1}{p!}\partial_i\mathcal{F}_{a_1a_2\cdots a_p}^{(p)} \Big(\zeta_{38}\Omega_a{}^{ai}\partial_b F_{a_0}^b + \zeta_{39}\Omega^{bai}\partial_b F_{aa_0} + \zeta_{40}\Omega_{a_0}{}^{bi}\partial_a F_b{}^a \Big) \Big]
\end{aligned} \tag{39}$$

where we have also imposed the bulk equations of motion (23). In above equation ζ_i with $i = 1, \dots, 40$ are the unknown constants that have to be found by consistency with dualities and with the S-matrix elements. One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the killing index y is a world volume index which is carried only by the RR field strength. This index can not be carried by the transverse scalar field. When it is carried by the field strength of the gauge field, the consistency with T-duality requires the couplings of one C_{p+1} and two transverse scalar fields which we consider them next.

5.3 One RR and two transverse scalar fields

All possible non-zero couplings of one RR potential C_{p+1} -form and two scalar fields after imposing the bulk equations of motion (23) are the following:

$$\begin{aligned}
S_{c\chi\chi} = & \frac{\pi^2\alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0a_1\cdots a_p} \Big[\frac{1}{(p+1)!} \partial_b \mathcal{F}_{ca_0a_1\cdots a_p}^{(p+2)} \Big(\rho_1 \Omega^{cai} \Omega_{ai}^b + \rho_2 \Omega_a{}^{ai} \Omega^{bc}{}_i \Big) \\
& - \frac{1}{(p+1)!} \partial^b \mathcal{F}_{ba_0a_1\cdots a_p}^{(p+2)} \Big(\rho_3 \Omega_a{}^{ai} \Omega_c{}^c{}_i + \rho_4 \Omega_{cai} \Omega^{cai} \Big) \\
& + \frac{1}{(p+1)!} \partial_j \mathcal{F}_{ia_0a_1\cdots a_p}^{(p+2)} \Big(\rho_5 \Omega_a{}^{ai} \Omega_c{}^{cj} + \rho_6 \Omega_{ca}{}^j \Omega^{cai} \Big) \\
& + \frac{1}{p!} \partial_b \mathcal{F}_{ija_1\cdots a_p}^{(p+2)} \Big(\rho_7 \Omega_{a_0}{}^{ai} \Omega_{a_1}^b{}_j + \rho_8 \Omega_a{}^{ai} \Omega_{a_0}^b{}_j \Big) + \frac{\rho_9}{p!} \Omega_{a_0}{}^a{}_i \Omega^{cb}{}_i \partial_c \mathcal{F}_{aba_1\cdots a_p}^{(p+2)} \\
& + \frac{1}{p!} \partial^c \mathcal{F}_{bca_1a_2\cdots a_p}^{(p+2)} \Big(\rho_{10} \Omega_{a_0}{}^{ai} \Omega_{ai}^b + \rho_{11} \Omega_a{}^{ai} \Omega_{a_0i}^b \Big) \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p!} \partial_i \mathcal{F}_{jba_1 a_2 \dots a_p}^{(p+2)} \left(\rho_{12} \Omega_{a_0}^{ai} \Omega_a^b \Omega_{a_0}^j + \rho_{13} \Omega_a^{ai} \Omega_{a_0}^b \Omega_{a_0}^j \right) \\
& + \frac{1}{p!} \partial_i \mathcal{F}_{jba_1 a_2 \dots a_p}^{(p+2)} \left(\rho_{14} \Omega_{a_0}^{aj} \Omega_a^b \Omega_{a_0}^i + \rho_{15} \Omega_a^{aj} \Omega_{a_0}^b \Omega_{a_0}^i \right) \\
& + \frac{1}{(p-1)!} \left(\rho_{16} \Omega_{a_1}^a \Omega_{a_0}^i \Omega_{a_0}^b \Omega_{a_0}^j \partial_a \mathcal{F}_{ijba_2 \dots a_p}^{(p+2)} + \rho_{17} \Omega_{a_1}^a \Omega_{a_0}^j \Omega_{a_0}^b \Omega_{a_0}^i \partial_j \mathcal{F}_{iaba_2 \dots a_p}^{(p+2)} \right) \\
& + \frac{1}{(p-1)!} \partial_{a_4} \mathcal{F}_{ijba_1 a_2 a_3 a_5 \dots a_p}^{(p+2)} \left(\rho_{18} \Omega_{a_0}^{ai} \Omega_a^b \Omega_{a_0}^j + \rho_{19} \Omega_a^{ai} \Omega_{a_0}^b \Omega_{a_0}^j \right) \\
& + \frac{\rho_{20}}{(p-1)!} \Omega_{a_1}^{ai} \Omega_{a_0 a}^j \partial^c \mathcal{F}_{ijca_2 \dots a_p}^{(p+2)} + \frac{\rho_{21}}{(p-1)!} \Omega_{a_1 i}^a \Omega_{a_0}^b \Omega_{a_0}^i \partial^c \mathcal{F}_{abca_2 \dots a_p}^{(p+2)} \Big]
\end{aligned} \tag{40}$$

where ρ_i with $i = 1, \dots, 21$ are the unknown constants. One can easily verify that the above couplings are consistent with the T-duality transformations (14) when the killing index y is a world volume index. So there is no D_p-brane coupling involving the RR potential C_{p+3} . The above couplings are also consistent with the S-duality of the D₃-brane action.

Now consider the sum of couplings (38), (39) and (40), *i.e.*,

$$S_p^{CS} = S_{caa} + S_{ca\chi} + S_{c\chi\chi} \tag{41}$$

They are not invariant under the linear T-duality transformations (14) for arbitrary coefficients. Imposing the invariance under T-duality (15), one finds the following relations between the constants in the CS part:

$$\begin{aligned}
\rho_2 &= -\rho_1, \quad \rho_9 = \rho_{10}, \quad \rho_3 = -\frac{\rho_1}{2}, \quad \rho_4 = \frac{\rho_1}{2}, \quad \rho_{11} = -\rho_{10}, \quad \rho_{15} = \zeta_{17} + \zeta_{18} - \zeta_{19} - \zeta_{20} \\
&+ \zeta_{21} + \zeta_{23} + \zeta_{24} - \zeta_{30} - \zeta_{31} + \zeta_{32} + \zeta_{33} - \zeta_{34} - \zeta_{36} - \zeta_{37} - \rho_{12} - \rho_{13} - \rho_{14} \\
\zeta_8 &= -\zeta_1 - \zeta_2 - \zeta_{17} - \zeta_{18} + \zeta_{19} + \zeta_{20} - \zeta_{21} - \zeta_{23} - \zeta_{24} - \zeta_3 + \zeta_4 - \zeta_6 + \zeta_7, \\
\kappa_6 &= \zeta_9 - \zeta_{10} - \kappa_1 + 2\kappa_9 + 2\kappa_{10} - 2\kappa_{11} + \kappa_{12} - \kappa_{13} - \kappa_{14} + \kappa_{15} + \kappa_{16} - \kappa_{17} + \kappa_3 - \kappa_{19} \\
&+ \kappa_4 + \kappa_{22} + \frac{1}{2}(\zeta_1 - \zeta_{11} + \zeta_{12} - \zeta_{13} + \zeta_{14} + \zeta_{18} - \zeta_{19} - \zeta_{21} - \zeta_{23} - \zeta_3 - \zeta_4 - \zeta_6), \\
\rho_{17} &= \frac{1}{2}(-\zeta_{11} + \zeta_{12} + \zeta_{13} - \zeta_{14} - \zeta_{18} + \zeta_{19} - \zeta_{21} - \zeta_{23} - 2\zeta_{28} + 2\zeta_{29} + \zeta_{31} - \zeta_{32} \\
&+ \zeta_{34} + \zeta_{36} + \rho_{12} + \rho_{14}), \quad \rho_{19} = \frac{1}{2}(\zeta_{11} - \zeta_{12} + \zeta_{13} - \zeta_{14} - \zeta_{18} + \zeta_{19} + 2\zeta_{20} \\
&- \zeta_{21} - \zeta_{23} - 2\zeta_{24} + \zeta_{31} - \zeta_{32} - 2\zeta_{33} + \zeta_{34} + \zeta_{36} + 2\zeta_{37} + \rho_{12} + 2\rho_{13} + \rho_{14} + 2\rho_{16}), \\
\rho_{20} &= \frac{1}{4}(-2\zeta_{10} - \zeta_{11} + \zeta_{12} - \zeta_{13} + \zeta_{14} + \zeta_{18} - \zeta_{19} - \zeta_{21} - \zeta_{23} - \zeta_{31} + \zeta_{32} + \zeta_{34} + \zeta_{36} \\
&+ 2\zeta_9 + \rho_{12} - \rho_{14} - 2\rho_{16} - 2\rho_{18}), \quad \rho_5 = \frac{1}{2}(\zeta_1 - 2\zeta_{25} + \zeta_3 + \zeta_{31} - \zeta_{32} + \zeta_{34} + \zeta_{36} + 2\zeta_{38} \\
&- \zeta_4 + \zeta_6 + \rho_{12} + \rho_{14}), \quad \rho_6 = \frac{1}{2}(-\zeta_1 - 2\zeta_{26} - \zeta_3 - \zeta_{31} + \zeta_{32} - \zeta_{34} - \zeta_{36} + 2\zeta_{39} + \zeta_4 - \zeta_6 \\
&- \rho_{12} - \rho_{14}), \quad \rho_7 = \frac{1}{2}(\zeta_1 + \zeta_3 - \zeta_{31} + \zeta_{32} + \zeta_{34} + \zeta_{36} + \zeta_4 + \zeta_6 + \rho_{12} - \rho_{14}), \quad \rho_8 = \frac{1}{2}(-\zeta_1
\end{aligned}$$

$$\begin{aligned}
& -2\zeta_{17} - 2\zeta_{18} + 2\zeta_{19} - 2\zeta_2 + 2\zeta_{20} - 2\zeta_{21} - 2\zeta_{23} - 2\zeta_{24} - \zeta_3 + \zeta_{31} - \zeta_{32} - 2\zeta_{33} + \zeta_{34} + \zeta_{36} \\
& + 2\zeta_{37} + \zeta_4 - \zeta_6 + \rho_{12} + 2\rho_{13} + \rho_{14}), \quad \kappa_8 = \kappa_{12} - \kappa_{13} - \kappa_{14} + \kappa_{15} - \kappa_{20} + \kappa_{23} + \kappa_7 - \zeta_2 \\
& - \zeta_{17} + \frac{1}{2}(-\zeta_1 - \zeta_{11} + \zeta_{12} - \zeta_{13} + \zeta_{14} - \zeta_{18} + \zeta_{19} - \zeta_{21} - \zeta_{23} - \zeta_3 + \zeta_4 - \zeta_6) \quad (42)
\end{aligned}$$

The above constraints make the CS action to be consistent with the T-duality. There are still many constants that are not fixed yet.

Imposing the constraints (42), one finds the couplings (39) are not consistent with S-duality for D₃-brane case. The S-duality requires, up to some total derivative terms, the couplings in (39) to be in the form of $\Omega\partial(*\mathcal{F}^T)\mathcal{M}_0\partial^2\mathcal{B}$. Using the expansion (20), one finds the following relation between the constants in the CS part and the DBI part:

$$\begin{aligned}
\zeta_4 &= 1 + \zeta_1 + \zeta_{13} - \zeta_{14} + \zeta_2 + \zeta_{17} - \zeta_{28} + \zeta_{29} + \zeta_{31} - \zeta_{32} \\
\zeta_5 &= 1 - 2\gamma_1 + \zeta_{13} - \zeta_{14} + \zeta_2 + \zeta_{17} + \zeta_3 + \zeta_{27} - \zeta_{28} + \zeta_{29} + \zeta_{34} - \zeta_{35} - \zeta_{40} \\
\zeta_6 &= 1 - \zeta_{11} + \zeta_{12} - \zeta_2 - \zeta_{17} - \zeta_{18} + \zeta_{19} - \zeta_{21} - \zeta_{23} - \zeta_3 - \zeta_{28} + \zeta_{29} + \zeta_{31} - \zeta_{32} \\
\zeta_9 &= \zeta_{10} + \zeta_{13} - \zeta_{14} - \zeta_{18} + \zeta_{19} - \zeta_{28} + \zeta_{29} + \zeta_{31} - \zeta_{32} \\
\zeta_{30} &= 1 - 2\gamma_1 + \zeta_{13} - \zeta_{14} + \zeta_{17} - \zeta_{28} + \zeta_{29} \\
\zeta_{36} &= -2 + 2\gamma_1 + \zeta_{11} - \zeta_{12} - \zeta_{13} + \zeta_{14} + \zeta_{18} - \zeta_{19} + \zeta_{21} + \zeta_{23} + 2\zeta_{28} - 2\zeta_{29} - \zeta_{31} + \zeta_{32} - \zeta_{34} \\
\zeta_{37} &= 1 - \gamma_3 - \zeta_{11} + \zeta_{12} - \zeta_{20} + \zeta_{24} - \zeta_{28} + \zeta_{29} + \zeta_{33} \\
\zeta_{38} &= 1 - 2\gamma_1 + \zeta_{13} - \zeta_{14} + \zeta_2 + \zeta_{17} + \zeta_{25} - \zeta_{28} + \zeta_{29} \\
\zeta_{39} &= -1 + 2\gamma_1 - \zeta_{13} + \zeta_{14} - \zeta_2 - \zeta_{17} + \zeta_{26} + \zeta_{28} - \zeta_{29} \quad (43)
\end{aligned}$$

as well as the constraint (34). Imposing the above constraints, one finds not only the couplings (39) but also the couplings (38) become consistent with the S-duality for D₃-brane, *i.e.*, the couplings $\partial\partial C_0\partial F\partial F$ in (38) and the couplings $\partial\partial\Phi\partial F\partial F$ in the DBI part combine into the S-duality invariant structure (20).

We now compare the couplings with the S-matrix elements. Imposing the constraints (42) and (43) into the action S_p^{CS} , one finds the resulting couplings are consistent with the S-matrix elements (22) provided that

$$\rho_{14} = -\rho_{12}, \quad \gamma_1 = 1 \quad (44)$$

The final step is to ignore the couplings which are total derivative terms or the couplings which can be eliminated by the Bianchi identities. Imposing the constraints (42), (43) and (44) into the action, we find the terms with coefficient ρ_{13} in (40) are total derivative terms, so ρ_{13} can be eliminated from the physical couplings. The terms with coefficients $\rho_1, \rho_{10}, \rho_{12}, \rho_{18}, \rho_{21}$ in (40) can be canceled by the Bianchi identity. When we write the field strengths in (39) in terms of corresponding potentials, we find the terms with coefficients ζ_i with $i = 1, 3, 7, 10, 20, 23, 24, 25, 26, 33$ disappear, so these constants can be eliminated from (39) by the Bianchi identity. Moreover, we find that terms with coefficients ζ_i with $i =$

2, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 27, 28, 29, 31, 32, 34, 35, 40 are total derivative terms. As a result, these terms can be ignored too. In the couplings (38), the constants κ_i with $i = 7, 10, 11, 20, 23$ can be ignored by the Bianchi identities and the constants κ_i with $i = 1, 2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22$ can be ignored by total derivative terms.

The final results for the CS part are the couplings which appear in (11), (13) and the following couplings:

$$S_{CS} \supset \frac{\pi^2 \alpha'^2 T_p}{12} \int d^{p+1}x \epsilon^{a_0 a_1 \dots a_p} \left[\frac{\gamma_3}{2!(p-1)!} \Omega_a^{ai} \partial^b F_{a_1 a_0} \partial_i \mathcal{F}_{ba_2 \dots a_p}^{(p)} + \frac{\gamma_3}{p!} \Omega_a^{ai} \Omega_b^{bj} \partial_j \mathcal{F}_{ia_0 \dots a_p}^{(p+2)} \right. \\ \left. - \frac{1 - \gamma_3}{(p-1)!} \Omega_a^{ai} \Omega_{a_0}^{bj} \partial_{a_3} \mathcal{F}_{ijba_1 a_2 a_4 \dots a_p}^{(p+2)} + \frac{1 - \gamma_3}{p!} \Omega_a^{ai} \Omega_{a_0}^{bj} \partial_b \mathcal{F}_{ija_1 \dots a_p}^{(p+2)} \right] \quad (45)$$

where we have also used the following identity in the second term:

$$(p+1) \epsilon^{a_0 a_1 \dots a_p} \Omega_{a_0}^{bj} \partial_j \mathcal{F}_{iba_1 a_2 \dots a_p}^{(p+2)} = \epsilon^{a_0 a_1 \dots a_p} \Omega_b^{bj} \partial_j \mathcal{F}_{ia_0 a_1 a_2 \dots a_p}^{(p+2)} \quad (46)$$

In proving the above identity we have used the totally antisymmetric property of the RR field strength which can be used to replace the world volume index b on the left-hand side by a_0 . Using similar relation and writing the RR field strength in terms of RR potential, one can prove the following identity:

$$-p \Omega_a^{ai} \Omega_{a_0}^{bj} \partial_{a_3} \mathcal{F}_{ijba_1 a_2 a_4 \dots a_p}^{(p+2)} + \Omega_a^{ai} \Omega_{a_0}^{bj} \partial_b \mathcal{F}_{ija_1 \dots a_p}^{(p+2)} = 0 \quad (47)$$

Using the above identity one finds that couplings in the second line of (45) are zero. The couplings in the first line of (45) are consistent with the linear T-duality, the S-duality and are zero when the scalar fields are on-shell. Note that the coupling in the first term for the case of D₃-brane can be written as S-dual multiplet because $\Omega_a^{ai} \partial^b F^{cd} \partial_i H_{bcd}^{(3)}$ is zero by the Bianchi identity of the gauge field strength. Therefore, the coefficient γ_3 can not be fixed by the linear dualities and by the S-matrix element of one closed and two open strings. It may be fixed by the open string pole of the S-matrix element of two closed strings and one open string at order α'^2 or by the contact terms of the S-matrix element of three closed strings. We expect the square of the second fundamental form appears in the world-volume curvatures as in (5), (6) and (8). The second fundamental forms in the second term of (45) can not be extended to the curvature (8), so we speculate the coefficient of this term to be zero, *i.e.*,

$$\gamma_3 = 0 \quad (48)$$

It would be interesting to analyze in details the S-matrix element of two closed strings and one open string or the S-matrix element of three closed strings to confirm the above relation.

Requiring the consistency of the D-brane effective action at order α'^2 with S-matrix and with the linear dualities, we have found the couplings of one NSNS and two NS states in the DBI part to be (10) and (12), and the couplings of one RR and two NS states in the CS part

to be (11) and (13). On the other hand, the D-brane effective action at order α'^2 should be invariant under supersymmetry and κ symmetry. It would be interesting to verify the above couplings to be consistent with the supersymmetry and κ symmetry.

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References

- [1] C. Bachas, Phys. Lett. B **374**, 37 (1996) [arXiv:hep-th/9511043].
- [2] M. R. Douglas, arXiv:hep-th/9512077.
- [3] C. P. Bachas, P. Bain and M. B. Green, JHEP **9905**, 011 (1999) [arXiv:hep-th/9903210].
- [4] M. R. Garousi and R. C. Myers, Nucl. Phys. B **475**, 193 (1996) [arXiv:hep-th/9603194].
- [5] A. Hashimoto and I. R. Klebanov, Phys. Lett. B **381**, 437 (1996) [arXiv:hep-th/9604065].
- [6] M. B. Green, J. A. Harvey and G. W. Moore, Class. Quant. Grav. **14**, 47 (1997) [arXiv:hep-th/9605033].
- [7] Y. K. Cheung and Z. Yin, Nucl. Phys. B **517**, 69 (1998) [arXiv:hep-th/9710206].
- [8] R. Minasian and G. W. Moore, JHEP **9711**, 002 (1997) [arXiv:hep-th/9710230].
- [9] M. R. Garousi, JHEP **1003**, 126 (2010) [arXiv:1002.0903 [hep-th]].
- [10] M. R. Garousi, Phys. Lett. B **701**, 465 (2011) [arXiv:1103.3121 [hep-th]].
- [11] M. R. Garousi, arXiv:1412.8131 [hep-th].
- [12] D. Robbins and Z. Wang, JHEP **1405**, 072 (2014) [arXiv:1401.4180 [hep-th]].
- [13] M. B. Green and M. Gutperle, Nucl. Phys. B **498**, 195 (1997) [hep-th/9701093].
- [14] J. T. Liu and R. Minasian, Nucl. Phys. B **874**, 413 (2013) [arXiv:1304.3137 [hep-th]].
- [15] T. Buscher, Phys. Lett. B **194** (1987) 59; B **201** (1988) 466.
- [16] P. Meessen and T. Ortin, Nucl. Phys. B **541**, 195 (1999) [arXiv:hep-th/9806120].
- [17] E. Bergshoeff, C. M. Hull and T. Ortin, Nucl. Phys. B **451**, 547 (1995) [arXiv:hep-th/9504081].

- [18] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos and P. K. Townsend, Nucl. Phys. B **470**, 113 (1996) [arXiv:hep-th/9601150].
- [19] S. F. Hassan, Nucl. Phys. B **568**, 145 (2000) [arXiv:hep-th/9907152].
- [20] M. R. Garousi, JHEP **1002**, 002 (2010) [arXiv:0911.0255 [hep-th]].
- [21] G. W. Gibbons and D. A. Rasheed, Phys. Lett. B **365**, 46 (1996) [hep-th/9509141].
- [22] A. A. Tseytlin, Nucl. Phys. B **469**, 51 (1996) [hep-th/9602064].
- [23] M. B. Green and M. Gutperle, Phys. Lett. B **377**, 28 (1996) [hep-th/9602077].
- [24] M. R. Garousi and R. C. Myers, Nucl. Phys. B **542**, 73 (1999) [hep-th/9809100].
- [25] T. Nutma, “xTras: a field-theory inspired xAct package for Mathematica,” arXiv:1308.3493 [cs.SC].